A Fixed Point Theorem Characterizing Metric Completeness

Dolhare U. P.[1] & Nalawade V. V.[2]
[1]Assistant Professor,Department of Mathematics, D. S. M. College, Jintur (M. S.), India
[2]Associate Professor, Department of Mathematics, S. G. R. G. Shinde College, Paranda, (M. S.), India

Received: May 06, 2018 Accepted: June 10, 2018

ABSTRACT *In this research paper we have proved a theorem that characterizes the completeness of the metric space under consideration. As a result of this characterization we obtain a corollary that presents us a criterion to decide the completeness of a metric space. The paper contains two theorems; one is the converse of the other.*

Keywords: *Contractions, Metric Completeness, Fixed Points, Countably Infinite Set, Closed Set.*
AMS Subject Classification: 47H10

1. Introduction

Completeness is one of the vital properties of metric spaces. One of the important aspects in terms of which metric completeness can be characterized is Fixed Points. Banach [8] proved that every contraction on a complete metric space has a unique fixed point. However the converse may not be true. This remarkable result is known as Banach Contraction Mapping Principle. But after the Banach Contraction Mapping Principle several results came up that exhibit both necessary and sufficient conditions for metric completeness. For instance Suzuki [9] stated and proved a very simple generalization of Banach contraction mapping principle that characterizes metric completeness. P. V. Subrahmanyam [7] proved that the Kannan fixed point theorem necessarily imply metric completeness. Hu [1] also showed that if every contraction on closed subsets of a metric space has a fixed point then the whole metric space is complete. Further Kirk [13] proved that a metric space is complete if and only if every Caristi mapping [2, 3] on the metric space has a fixed point. We can see the many results in this direction in [4, 5, 10, 11]. We have defined a mapping on a metric space and proved that the metric space is complete if and only if it every such mapping has a unique fixed point.

2. Preliminaries and Definitions

Definition 2.1: Let (X,d) be a metric space and let T be a mapping on X. Then T is called a "Contraction" if there exists $r\in[0,1)$ such that $d(Tx,Ty)\le rd(x,y)$ for all $x,y\in X$.

The following famous theorem is referred to as the Banach contraction principle.

Theorem 2.1 (Banach) [8]: Let (X,d) be a complete metric space and let T be a contraction on X. Then T has a unique fixed point.

Definition 2.2: Let (X,d) be a metric space and let T be a mapping on X. Then T is called "Kannan" if there exists $r\in[0,1/2)$ such that $d(Tx,Ty)\le rd(x,Tx)+rd(y,Ty)$ for all $x,y\in X$.

The following result is due to P. V. Subrahmanyam [7].

Theorem 2.2 [7]: A metric space (X,d) in which every mapping T of X into itself, satisfies the conditions:

(1) $d(Tx,Ty)\le\lambda\left[d(x,Tx)+d(y,Ty)\right]$ for all $x,y\in X$, where λ is any positive number not necessarily less than 1/2;

(2) $T(X)$ is countable;

has a fixed point, is complete.

P. V. Subrahmanyam [7] also proved that the condition (1) in the theorem 2.2 can be replaced by any of the following conditions.

($1'$) $d(Tx,Ty)\le\lambda\left[d(x,Ty)+d(y,Tx)\right]$ for all $x,y\in X$ for a fixed $\lambda>0$, $0<\lambda<1/2$.

($1''$) $d(Tx,Ty)\le\lambda\max\left\{d(x,Ty),d(y,Tx)\right\}$ for all $x,y\in X$ for a fixed $\lambda>0$.

($1'''$) $d(Tx,Ty)\le\lambda\max\left\{d(x,Tx),d(y,Ty)\right\}$ for all $x,y\in X$ for a fixed $\lambda>0$.

Theorem 2.3 (Kikkawa and Suzuki) [6]: Let T be a mapping on complete metric space (X,d) and θ be a non-increasing function from $[0,1)$ onto $(1/2,1]$ defined by

$$\theta(r)=\begin{cases}1, \text{if } 0\le r\le \dfrac{\sqrt{5}-1}{2}, \\ \dfrac{1-r}{r^2}, \text{if } \dfrac{\sqrt{5}-1}{2}\le r\le \dfrac{1}{\sqrt{2}}, \\ \dfrac{1}{1+r}, \text{if } \dfrac{1}{\sqrt{2}}\le r<1\end{cases}$$

Suppose that there exists $r\in[0,1)$ such that $\theta(r)d(x,Tx)\le d(x,y)$ implies $d(Tx,Ty)\le r\max\{d(x,Tx),d(y,Ty)\}$ for all $x,y\in X$. Then T has a unique fixed point z and $\lim_{n\to\infty}T^n x=z$ holds for every $x\in X$.

The following is a Kannan version of the Suzuki Theorem.

Theorem 2.4 (Kikkawa and Suzuki) [6]: Let T be a mapping on complete metric space (X,d) and θ be a non-increasing function from $[0,1)$ into $(1/2,1]$ defined by

$$\theta(r)=\begin{cases}1, \text{if } 0\le r\le \dfrac{1}{\sqrt{2}}, \\ \dfrac{1}{1+r}, \text{if } \dfrac{1}{\sqrt{2}}\le r<1\end{cases}$$

Let $\alpha\in[0,1/2)$ and $r=\dfrac{\alpha}{1-\alpha}\in[0,1)$.

Suppose that $\theta(r)d(x,Tx)\le d(x,y)$ implies $d(Tx,Ty)\le \alpha d(x,Tx)+\alpha d(y,Ty)$ for all $x,y\in X$. Then T has a unique fixed point z and $\lim_{n\to\infty}T^n x=z$ holds for every $x\in X$.

Recently Vidyadhar V. Nalawade and Uttam P. Dolhare [12] have proved another variant of the above result as follows.

Theorem 2.5 [12]: Let (X,d) be a complete metric space and let T be a mapping on X. Define a non-increasing function θ from $[0,1)$ into $(1/2,1]$ by

$$\theta(r)=\begin{cases}1, \text{if } 0\le r\le \dfrac{\sqrt{5}-1}{2}, \\ \dfrac{1-r}{r^2}, \text{if } \dfrac{\sqrt{5}-1}{2}\le r\le \dfrac{1}{\sqrt{2}}, \\ \dfrac{1}{1+r}, \text{if } \dfrac{1}{\sqrt{2}}\le r<1\end{cases}$$

Let $\alpha\in[0,1/2)$ and $r=\dfrac{\alpha}{1-\alpha}\in[0,1)$. Assume that $\theta(r)d(x,Tx)\le d(x,y)$ implies $d(Tx,Ty)\le \alpha d(x,Tx)+\alpha d(y,Ty)$ for all $x,y\in X$. Then T has a unique fixed point z and $\lim_{n\to\infty}T^n x=z$ holds for every $x\in X$.

Theorem 2.2 (Suzuki) [9]: Define a non-increasing function θ from $[0,1)$ to $(1/2,1]$ by

$$\theta(r)=\begin{cases}1, \text{if } 0\le r\le \dfrac{\sqrt{5}-1}{2},\\ \dfrac{1-r}{r^2}, \text{if } \dfrac{\sqrt{5}-1}{2}\le r\le \dfrac{1}{\sqrt{2}},\\ \dfrac{1}{1+r}, \text{if } \dfrac{1}{\sqrt{2}}\le r<1\end{cases}$$

Then for a metric space (X,d), the following are equivalent:

(1) X is complete.

(2) Every mapping T on X satisfying the following has a fixed point: there exists $r\in[0,1)$ such that $\theta(r)d(x,Tx)\le d(x,y)$ implies $d(Tx,Ty)\le rd(x,y)$ for all $x,y\in X$.

3. MAIN RESULTS

Throughout the following text we denote by $\mathbb{N}$ the set of all natural numbers.

Theorem 3.1: Let (X,d) be a complete metric space and let T be a mapping on X. Let $r\in[0,3/10)$. Assume that $d(x,Tx)\le d(x,y)$ implies $d(Tx,Ty)\le rd(x,Tx)+rd(y,Ty)$ for all $x,y\in X$. Then T has a unique fixed point z and $\lim_{n\to\infty}T^n x=z$ holds for every $x\in X$.

Proof: We have $d(x,Tx)\le d(x,Tx)$, for all $x\in X$.

By hypothesis we have,

$$d(Tx,T^2x)\le rd(x,Tx)+rd(Tx,T^2x)$$

$$\therefore d(Tx,T^2x)\le \frac{r}{1-r}d(x,Tx), \text{for all } x\in X.$$

In general

$$d(T^n x,T^{n+1}x)\le\left(\frac{r}{1-r}\right)^n d(x,Tx) \text{ for all } x\in X \text{..........(1)}$$

We now fix an element $u\in X$ and define a sequence $\{u_n\}_{n=1}^{\infty}$ in X by $u_n=T^n u$.

Then from (1) we have,

$$d(u_n,u_{n+1})=d(T^n u,T^{n+1}u)\le\left(\frac{r}{1-r}\right)^n d(u,Tu).$$

Taking limit as $n\to\infty$, since $\frac{r}{1-r}<1$ for $r\in[0,3/10)$, we get $d(u_n,u_{n+1})\to 0$ as $n\to\infty$.

Thus $\{u_n\}$ is a Cauchy sequence in X.

Since X is a complete metric space, $\{u_n\}$ converges to some point $z\in X$.

We next show

$$d(Tx,z)\le\left(\frac{r}{1-r}\right)d(x,z), \text{for all } x\in X\setminus\{z\} \text{..........(2)}$$

For any $x\in X\setminus\{z\}$, we have $d(u_n,z)\le\frac{d(x,z)}{3}$, for all $n\ge\nu$ for some $\nu\in\mathbb{N}$.

Then we must have

$$\begin{aligned} d(u_n, Tu_n) &\le d(u_n, Tu_n) = d(u_n, u_{n+1}) \\ &\le d(u_n, z) + d(u_{n+1}, z) \\ &\le \frac{d(x,z)}{3} + \frac{d(x,z)}{3}, \qquad \text{for all } n > \nu \\ &= \frac{2}{3} d(x,z) = d(x,z) - \frac{d(x,z)}{3} \\ &\le d(x,z) - d(u_n, z) \le d(u_n, x) \qquad (\because |d(x,z) - d(z,y)| \le d(x,y)) \end{aligned}$$

Thus $d(u_n, Tu_n) \le d(u_n, x)$.

By hypothesis we then get

$$d(Tu_n, Tx) \le rd(u_n, Tu_n) + rd(x, Tx), \text{for } n \ge \nu \text{...(3)} .$$

Taking n tend to ∞ we get

$$d(z, Tx) \le rd(z,z) + rd(x, Tx) = rd(x, Tx)$$

$$\therefore d(z, Tx) \le rd(x, Tx) \le r[d(x,z) + d(Tx, z)] \quad \text{(Triangle inequality)}$$

Thus

$$d(Tx, z) \le rd(x,z) + rd(Tx, z)$$

$$\therefore (1-r) d(Tx, z) \le rd(x, z)$$

$$\therefore d(Tx, z) \le \frac{r}{1-r} d(x,z)$$

Thus we have shown (2).

Next, we shall show that there exists a $j \in \mathbb{N}$ such that $T^j z = z$.

This we show by contradiction method. We assume that $T^j z \ne z$ for all $j \in \mathbb{N}$.

As $T^j z \ne z$, for all $j \in \mathbb{N}$ we can use inequality (2) for $T^j z$. (This is because inequality (2) is true for all $x \ne z$).

Using inequality (2) we get

$$d(T^2 z, z) = d(T \circ Tz, z) \le \left(\frac{r}{1-r}\right) d(Tz, z) \qquad (\text{taking } x = Tz \text{ in } (2))\text{..............................(4)}$$

$$d(T^3 z, z) = d(T \circ T^2 z, z) \le \left(\frac{r}{1-r}\right) d(T^2 z, z) \quad (\text{taking } x = T^2 z \text{ in } (2))$$

$$\le \left(\frac{r}{1-r}\right)^2 d(Tz, z) \qquad (\text{by } (4))\text{...(5)}$$

$$d(T^4 z, z) = d(T \circ T^3 z, z) \le \left(\frac{r}{1-r}\right) d(T^3 z, z) \quad (\text{taking } x = T^3 z \text{ in } (2))$$

$$\le \left(\frac{r}{1-r}\right)^3 d(Tz, z) \qquad (\text{by } (5)).$$

Thus in general

$$d(T^{j+1} z, z) \le \left(\frac{r}{1-r}\right)^j d(Tz, z) \text{ for any } j \in \mathbb{N} \text{ ....(6)} .$$

Now, $\frac{r}{(1-r)^2} < 1$ if $r \in [0, 3/10)$

Assume $d(T^2z,z) < d(T^2z,T^3z)$.
Then we have

$$d(z,Tz) \le d(z,T^2z) + d(Tz,T^2z) \qquad \text{(by triangle inequality)}$$
$$< d(T^2z,T^3z) + rd(z,Tz) \qquad \text{(by above assumption and (1))}$$
$$\le \left(\frac{r}{1-r}\right)^2 d(z,Tz) + \left(\frac{r}{1-r}\right) d(z,Tz) \qquad \text{(by (1))}$$
$$= \left(\frac{r^2}{(1-r)^2} + \frac{r}{1-r}\right) d(z,Tz)$$
$$= \left(\frac{r^2 + r(1-r)}{(1-r)^2}\right) d(z,Tz)$$
$$= \left(\frac{r}{(1-r)^2}\right) d(z,Tz)$$
$$< d(z,Tz) \qquad \left(\because \frac{r}{(1-r)^2} < 1\right)$$

Thus $d(z,Tz) < d(z,Tz)$. This is a contradiction.

So we have $d(T^2z,z) \ge d(T^2z,T^3z)$.

By hypothesis $d(T^3z,Tz) \le rd(T^2z,T^3z) + rd(z,Tz)$..(7)

Now consider

$$d(z,Tz) \le d(z,T^3z) + d(T^3z,Tz) \qquad \text{(Triangle inequality)}$$
$$\le \left(\frac{r}{1-r}\right)^2 d(z,Tz) + rd(T^2z,T^3z) + rd(z,Tz) \qquad \text{(by (6) and (7))}$$
$$\le \left(\frac{r}{1-r}\right)^2 d(z,Tz) + r\left(\frac{r}{1-r}\right)^2 d(z,Tz) + rd(z,Tz) \qquad \text{(by (1))}$$
$$= \left(\frac{r^2 + r^3 + r(1-\mathrm{r})^2}{(1-\mathrm{r})^2}\right) d(z,Tz)$$
$$= \left(\frac{2r^3 - r^2 + r}{(1-\mathrm{r})^2}\right) d(z,Tz)$$
$$< d(z,Tz) \qquad \left(\because \frac{2r^3 - r^2 + r}{(1-\mathrm{r})^2} < 1 \text{ if } 0 \le r < 3/10\right)$$

Thus $d(z,Tz) < d(z,Tz)$. This is a contradiction.

Therefore there exists a $j \in$ □ such that $T^jz = z$. Since $\{T^nz\}$ is a Cauchy sequence, we have $Tz = z$.

Thus z is fixed point of T.

Uniqueness: Let z' be another fixed point of T. Thus $Tz' = z'$. From (2) we get

$$d(z',z) = d(Tz',z) \le \left(\frac{r}{1-r}\right) d(z',z)$$

Thus $d(z',z) \le \left(\frac{r}{1-r}\right) d(z',z)$.

Therefore $d(z',z) = 0 \quad \left(\because \frac{r}{1-r} < 1\right)$ and so $z' = z$. Hence the fixed point of T is unique. The proof is complete.

Now we shall prove that every mapping T on a metric space X satisfying the condition in the theorem 3.1 is complete in the following theorem.

Theorem 3.2: Let (X,d) be a metric space. For $r \in [0,3/10)$, let F_1 be the family of mappings T on X satisfying the following.

For $x, y \in X, d(x,Tx) \le d(x,y)$ implies $d(Tx,Ty) \le rd(x,Tx) + rd(y,Ty)$.

Let F_2 be the family of mappings T on X satisfying a) and the following.

$T(X)$ is countably infinite.

Every subset of $T(X)$ is closed.

Then the following are equivalent:

1) The metric space X is complete.
2) Every mapping $T \in F_1$ has a fixed point for all $r \in [0,3/10)$.
3) There exists $r \in (0,3/10)$ such that every mapping $T \in F_2$ has a fixed point.

Proof: We shall prove 1) implies 2), 2) implies 3) and finally 3) implies 1).

1) Implies 2) is proved by the theorem 3.1. As $F_2 \subset F_1$ for all $r \in [0,3/10)$ it is also clear that 2) implies 3). Now we shall proceed to show 3) implies 1). This we shall show by contradiction method. So suppose that 3) is true but 1) is not. That is suppose the metric space (X,d) is not complete. Then there exists a Cauchy sequence $\{s_n\}_{n=1}^{\infty}$ in X that do not converge in X.

Define a function $f : X \to [0,\infty)$ by $f(x) = \lim_{n\to\infty} d(x,s_n)$ for all $x \in X$.

We shall show that for every $x \in X$, the sequence $\{d(x,s_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in the metric space $[0,\infty)$ with respect to the absolute value metric. We have $f(x) = \lim_{n\to\infty} d(x,s_n)$. So given $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|d(x,s_n) - f(x)| < \frac{\varepsilon}{2}$ for all $n \ge N$.

Consider

$$\begin{aligned}
|d(x,s_n) - d(x,s_m)| &\le \left|[d(x,s_n) - f(x)] - [d(x,s_m) - f(x)]\right| \\
&\le |d(x,s_n) - f(x)| + |(-1)[d(x,s_m) - f(x)]| \\
&= |d(x,s_n) - f(x)| + |-1||d(x,s_m) - f(x)| \\
&= |d(x,s_n) - f(x)| + |d(x,s_m) - f(x)| \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \qquad \text{for all } n,m \ge N \\
&= \varepsilon.
\end{aligned}$$

Thus $|d(x,s_n) - d(x,s_m)| < \varepsilon$ for all $n,m \ge N$. Hence the sequence $\{d(x,s_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in the metric space $[0,\infty)$. An immediate consequence of this result is that the function f defined above is well defined. Next we claim the followings:

[1] $|f(x) - f(y)| \le d(x,y)$.

Consider

$$|f(x)-f(y)| = \left|\lim_{n\to\infty} d(x,s_n) - \lim_{n\to\infty} d(y,s_n)\right|$$
$$= \left|\lim_{n\to\infty}\left[d(x,s_n)-d(y,s_n)\right]\right|$$
$$\le \left|\lim_{n\to\infty} d(x,y)\right| \qquad (\because |d(x,z)-d(y,z)| \le d(x,y))$$
$$= |d(x,y)|$$
$$= d(x,y) \qquad (\because d(x,y)\ge 0)$$

[2] $d(x,y) \le f(x)+f(y)$.

Consider

$$d(x,y) \le d(x,s_n)+d(y,s_n)$$
$$\le \lim_{n\to\infty}\left[d(x,s_n)+d(y,s_n)\right]$$
$$= \lim_{n\to\infty} d(x,s_n) + \lim_{n\to\infty} d(y,s_n)$$
$$= f(x)+f(y)$$

[3] $f(x) > 0, \text{for all } x \in X.$

It is clear that, $\lim_{n\to\infty} d(x,s_n) \ge 0$ because $d(x,s_n)\ge 0 \text{ for all } x\in X$. But if $\lim_{n\to\infty} d(x,s_n) = 0$, then $\lim_{n\to\infty} s_n = x \in X$, which is a contradiction to the hypothesis that the sequence $\{s_n\}_{n=1}^{\infty}$ do not converge in X. Therefore $\lim_{n\to\infty} d(x,s_n) > 0, \text{for all } x\in X$ or in other words $f(x) > 0, \text{for all } x \in X$.

[4] $\lim_{n\to\infty} f(s_n) = 0$.

Consider

$$\lim_{n\to\infty} f(s_n)$$
$$= \lim_{n\to\infty}\left[\lim_{m\to\infty} d(s_n,s_m)\right]$$
$$= \lim_{n\to\infty}[0] \qquad (\text{because } \{s_n\}_{n=1}^{\infty} \text{ is a Cauchy sequence})$$
$$= 0$$

Define the mapping $T: X \to X$ as follows: Since $\lim_{n\to\infty} f(s_n) = 0$ and since $f(x) > 0$ for each $x \in X$ there exists $\nu \in \mathbb{N}$ such that $f(s_\nu) \le \frac{\alpha}{3+\alpha} f(x)$, where $0 \le \alpha < \frac{3}{13}$. We define $Tx = s_\nu$.

It is then obvious that $f(Tx) = f(s_\nu) \le \frac{\alpha}{3+\alpha} f(x) < f(x)$ for all $x \in X$. Then $Tx \ne x$ for all $x \in X$. This is because if $Tx = x$ then $f(Tx) = f(x)$ for all $x \in X$, but $f(Tx) < f(x)$. Thus T does not have a fixed point.

We shall show that T satisfies (a), (b) and (c) of the statement of the theorem.

Fix $x, y \in X$ satisfying $d(x,Tx) \le d(x,y)$.

Case 1. $f(y) > 2f(x)$.

$$
\begin{aligned}
d(Tx,Ty) &\le f(Tx)+f(Ty) && \text{(by[2]above)}\\
&\le \frac{\alpha}{3+\alpha}(f(x)+f(y)\\
&\le \frac{\alpha}{3}(f(x)+f(y)) && \left(\text{because } 3<3+\alpha \Rightarrow \frac{1}{3+\alpha}<\frac{1}{3}\right)\\
&\le \frac{\alpha}{3}(f(x)+f(y))+\frac{2\alpha}{3}(f(y)-2f(x)) && (\text{because } f(y)>2f(x) \text{ in this case})\\
&= \alpha(f(y)-f(x))\\
&\le \alpha d(x,y) && \text{(by[1]above)}\\
&\le \alpha\left[d(x,Tx)+d(Tx,y)\right] && \text{(by Triangle Inequality)}\\
&= \alpha d(x,Tx)+\alpha d(Tx,y)\\
&\le \alpha d(x,Tx)+\alpha\left[d(y,Ty)+d(Ty,Tx)\right] && \text{(by Triangle Inequality)}\\
&= \alpha d(x,Tx)+\alpha d(y,Ty)+\alpha d(Tx,Ty)
\end{aligned}
$$

Thus

$$d(Tx,Ty) \le \alpha d(x,Tx)+\alpha d(y,Ty)+\alpha d(Tx,Ty)$$

$$\therefore (1-\alpha)d(Tx,Ty) \le \alpha d(x,Tx)+\alpha d(y,Ty)$$

$$\therefore d(Tx,Ty) \le \frac{\alpha}{1-\alpha}d(x,Tx)+\frac{\alpha}{1-\alpha}d(y,Ty)$$

$$\therefore d(Tx,Ty) \le rd(x,Tx)+rd(y,Ty) \qquad \left(\text{Because as } 0\le\alpha<\frac{3}{13}, 0\le\frac{\alpha}{1-\alpha}<\frac{3}{10} \text{ or } 0\le \mathrm{r}<\frac{3}{10}\right)$$

Case 2. $f(y)\le 2f(x)$.

We observe that

$$
\begin{aligned}
d(x,y) &\ge d(x,Tx) && \text{(by assumption)}\\
&\ge f(x)-f(Tx) && \text{(by[1]above)}\\
&\ge \left(1-\frac{\alpha}{3+\alpha}\right)f(x) && \left(f(Tx)\le\frac{\alpha}{3+\alpha}f(x)\Rightarrow -f(Tx)\ge-\frac{\alpha}{3+\alpha}f(x)\right)\\
&= \frac{3}{3+\alpha}f(x)
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
d(Tx,Ty) &\le f(Tx)+f(Ty)\\
&\le \frac{\alpha}{3+\alpha}(f(x)+f(y))\\
&\le \frac{\alpha}{3+\alpha}(f(x)+2f(x)) && (\text{because } f(y)\le 2f(x) \text{ in this case})\\
&= \frac{3\alpha}{3+\alpha}f(x) && \left(\text{because } \frac{3}{3+\alpha}f(x)\le d(x,y)\right)\\
&\le \alpha d(x,y)
\end{aligned}
$$

Then continuing as in case 1 we reach to the conclusion that

$d(Tx,Ty)\le rd(x,Tx)+rd(y,Ty)$.

Thus we have shown (a).

Also since $Tx = s_v \in \{s_n / n \in \mathbb{N}\}$ it is clear that $T(X) \subset \{s_n / n \in \mathbb{N}\}$. Hence $T(X)$ is countably infinite. Hence (b) is true. It is not difficult to show (c).

Thus T satisfies (a), (b) and (c) in the statement of the theorem. So $T \in F_2$. By (3) in the statement of the theorem, T has a fixed point. This is a contradiction. This contradiction yields the theorem.

Consequence of Theorem 3.2 is the following corollary.

Corollary 3.1: For a metric space (X, d) the following are equivalent:

(1) X is complete,

(2) There exists $r \in (0, 3/10)$ such that every mapping T on X satisfying the following condition has a fixed point: $d(x,Tx) \le d(x,y)$ implies $d(Tx,Ty) \le rd(x,Tx) + rd(y,Ty)$ for all $x, y \in X$.

ACKNOWLEDGEMENT: Authors are thankful to the referees for their valuable suggestions.

REFERENCES

[1] Hu, T. K.: On a fixed-point theorem for metric spaces. Amer. Math. Monthly 74, 436-437 (1967).

[2] J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, Trans. Amer. Math. Soc., 215 (1976), 241–251. MR0394329 (52:15132).

[3] J. Caristi and W. A. Kirk, Geometric fixed point theory and inwardness conditions, Lecture Notes in Math., Vol. 490, pp. 74–83, Springer, Berlin, 1975. MR0399968 (53:3806).

[4] J. Dugundji, Positive definite functions and coincidences, Fund. Math., 90 (1976), 131–142. MR0400192 (53:4027).

[5] J. D. Weston, A characterization of metric completeness, Proc. Amer. Math. Soc., 64 (1977), 186–188. MR0458359 (56:16562).

[6] Kikkawa M., Suzuki T., Some similarity between contractions and Kannan mappings, Fixed Point Theory Appl., 2008, Article ID 649749, 1–8.

[7] P. V. Subrahmanyam, Completeness and fixed-points, Monatsh. Math., 80 (1975), 325–330. MR0391065 (52:11887).

[8] S. Banach, Sur les op´erationsdans les ensembles abstraitsetleur application aux equations int´egrales, Fund. Math., 3 (1922), 133–181.

[9] Suzuki T., A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc., 2008, 136, 1861–1869.

[10] S. Park, Characterizations of metric completeness, Colloq. Math., 49 (1984), 21–26. MR774845 (86d:54042).

[11] S. Reich, Kannan's fixed point theorem, Boll. Un. Mat. Ital., 4 (1971), 1–11. MR0305163 (46:4293).

[12] Vidyadhar V. Nalawade, Uttam P. Dolhare, Another Kannan Version of Suzuki Fixed Point Theorem, International Journal of Mathematics Trends and Technology (IJMTT), Volume 36, Number 2, August 2016.

[13] W. A. Kirk, Caristi's fixed point theorem and metric convexity, Colloq. Math., 36 (1976), 81–86. MR0436111 (55:9061).